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Boundary Variational Formulation of a Frictionless Contact Problem for Composite Finite Bodies

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In this paper the frictionless contact problem for bonded, dissimilar homogeneous elastic three-dimensional bodies is addressed. A new method, based on a rigorous variational formulation and application of a direct multiregional boundary element procedure for the approximation of the related Green's function, has been developed. The method is applicable to any kind of geometry of the contacting bodies and for arbitrary loading. The examples presented illustrate a distinct ability of the method to capture the influence of the composite structure of a body on the contact area and the pressure acting in it.

Nomenclature

A	= operator from a space V into its dual V'
$a(u, v)$	= continuous bilinear form on $V \times V$ representing the virtual work in an elastic body Ω
E_{ijkm}	= components of Hooke's tensor; the elasticities of the material
$f(v)$	= continuous linear functional on V representing the work done by applied body forces f and surface tractions t on the virtual displacement v; the generalized forces
G	$= A^{-1}$
$G_{ij}(x, \eta)$	= Green's matrix
g	= gap function between the bodies in contact as measured along the normal n before deformation
$H^m(\Omega)$	= Hilbert space
K	= subset of V consisting of displacements which satisfy the contact condition
n	= unit outward normal to Γ
p	= pressure along the contact surface
t	= traction vectors
$T_{ij}(x, \eta)$	= fundamental solution, in terms of traction, for the isotropic three-dimensional elastostatic problem
$U_{ij}(x, \eta)$	= fundamental solution, in terms of displacement, for the isotropic three-dimensional elastostatic problem
$u(x)$	= displacement vector at a material point x in the body Ω
$v(x)$	= arbitrary admissible displacement vector in the body Ω
V	= space of admissible displacements on which the virtual work is well-defined
V'	= dual space of a normed linear space V; the space of continuous linear functionals over V
$\Gamma \equiv \partial\Omega$	= boundary surface of a body Ω
Γ_D, Γ_F	= portions of boundary where the displacements and tractions are specified, respectively
Γ_C	= candidate for the contact surface
Γ_I^{ab}, Γ_I^{cd}	= internal boundaries between regions a and b of body 1, and regions c and d of body 2, respectively.
Ω	= open, bounded, simply connected domain occupied by a linear elastic body
γ	= trace operator from $H^m(\Omega)$ into $H^{m-1/2}(\Gamma)$
σ_{ij}	= stress tensor
σ_n	$= \sigma_{ij} n_i n_j$
σ_{T_i}	$= \sigma_{ij} n_j - \sigma_n n_i$

Subscripts

n	= normal component

Superscript

R	= relative value
x	= a, b, c, or d parts of composite bodies 1 and 2
α	= composite bodies 1 or 2

I. Introduction

In recent years, much interest has been devoted to the mathematical formulation of structural problems involving unilateral constraints (Panagiotopoulos[1]; Kikuchi and Oden[2]; and Alliney et al.[3]). An important class of these problems is the frictionless contact between two elastic composite bodies. More specifically, given the geometry of both bodies, their elastic properties, and the total compressive force, the objective is to find the contact area and the pressure acting in it. Because the contact area is a priori unknown, unilateral conditions must be imposed on the relative displacement and the mutual reactions.

The first successful attempt to analyze such contacts was made by Hertz.[4] For a limited class of geometries of which the half-space is a pre-eminent member, exact solutions via integral equations are possible. An excellent treatise on the analysis of contact problems by classical methods is given by Gladwell.[5] For more general shapes the usual finite element method in connection with a suitable iteration for the contact pressure may be employed (Chan and Tuba[6]). A rigorous numerical treatment of contact problems, however, has to start from the variational inequalities as shown in Fichera[7] and Kikuchi.[8]

The first variational formulations defined on the contact area were proposed in the seventies. A variety of variational principles for contact problems and an extensive bibliography is given in Kalker.[9] A systematic and rigorous mathematical treatment of the variational boundary formulations in terms of the Green's function is reported in Kikuchi and Oden.[2] A class of contact problems with an unknown contact zone for the laminated half-space was analyzed by Bufler et al.[10] Recently, Panagiotopoulos and Lazaridis,[11] and Alliney et al.[3] have proposed a numerical scheme similar to the one pre-

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sented in this paper for the contact problem with unilateral constraints for two-dimensional homogeneous bodies.

To sum up, analytical solutions to date lack the ability to deal with complex three-dimensional problems, while the finite element method lacks the rigor of the variational approach. The last limitation is a consequence of the fact that the finite element method approximates the internal equilibrium solution or internal variables of the problem. Our proposed methodology, based on the variational approach and the multiregional direct boundary element method, alleviates both of these problems. In a previous paper[12] we presented a numerical method for homogeneous, two-body, frictionless contact problems. Presented here is an extension of that work to nonhomogeneous contact problems.

II. Formulation of the Problem

Two elastic composite bodies, Ω^1 $(=\Omega^a \cup \Omega^b)$ and Ω^2 $(=\Omega^c \cup \Omega^d)$, are pressed together as in Fig. 1. The contact area Γ_C^α is assumed to be small so that only displacements perpendicular to the common tangential plane at a representative contact point have to be taken into account. Furthermore, the two surfaces Γ_C^α $(\alpha = 1, 2)$ can be identified with their projection Γ_C onto the tangential (contact) plane.

In the present formulation we have assumed that the contact surface Γ_C has an empty intersection with both Γ_I^{ab} and Γ_I^{cd}. This assumption circumvents the complications that the stress singularity at the free edge of the interface would introduce in the contact problem.

Assuming that the constituent bodies Ω^x are in equilibrium, the problem becomes one of finding displacement fields u_i^x, stress fields σ_{ij}^x, and the contact pressure p that satisfy the equilibrium conditions

$$-(E_{ijkm}u_{m,k})_{,j}^x = 0 \text{ in } \Omega^x \tag{1}$$

We assume that the elasticity coefficients E_{ijkm}^x in Eq. (1) satisfy the usual symmetry and ellipticity conditions.

The associated natural and essential conditions, imposed separately on the external boundaries Γ^x, are given as

$$t_i^x = (t_i^x)_0 \text{ on } \Gamma_F^x \tag{2}$$

$$u_i^x = 0 \text{ on } \Gamma_D^x \tag{3}$$

The interface conditions on Γ_I^{ab} and Γ_I^{cd} are given as

$$\left.\begin{aligned} u_i^a &= u_i^b \\ t_i^a &= -t_i^b \end{aligned}\right\} \text{on } \Gamma_I^{ab} \tag{4}$$

$$\left.\begin{aligned} u_i^c &= u_i^d \\ t_i^c &= -t_i^d \end{aligned}\right\} \text{on } \Gamma_I^{cd} \tag{5}$$

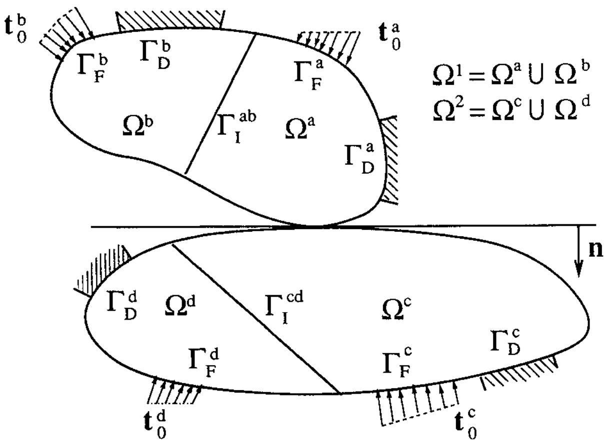

Fig. 1 Model of two-composite-body contact problems.

Finally, the frictionless contact condition is given by

$$\left.\begin{aligned} &\sigma_{T_i} = 0 \\ &p = 0 \qquad \text{for} \qquad u_n^R < g \\ &p \le 0 \qquad \text{for} \qquad u_n^R = g \end{aligned}\right\} \text{on } \Gamma_C \tag{6}$$

Because the contact zone is not known in advance, the problem is nonlinear.

To set the stage for the reciprocal variational formulation we emphasize the reciprocal structure of the contact condition (6); i.e.,

$$(q - p)(u_n^R - g) \ge 0 \qquad \forall\, q \text{ s.t. } q \le 0 \tag{7}$$

on Γ_C.

III. Reciprocal Variational Formulation for Two-Composite-Body Contact Problems

We shall now introduce a reciprocal variational principle for the problem considered. Let the domain $\Omega^\alpha \in \mathcal{C}^{1,1}$. We begin by introducing the space of admissible displacements v^α of a linearly elastic body $\bar{\Omega}^\alpha$ $(\alpha = 1, 2)$

$$\begin{aligned} V^1 = \{ v^1 = (v^a, v^b) \in H^1(\Omega^a) \times H^1(\Omega^b) \mid \\ \gamma_D^a(v^a) = 0 \text{ in } H^{1/2}(\Gamma_D^a) \\ \gamma_D^b(v^b) = 0 \text{ in } H^{1/2}(\Gamma_D^b) \\ \gamma_I^a(v^a) = \gamma_I^b(v^b) \text{ in } H^{1/2}(\Gamma_I^{ab}) \} \end{aligned} \tag{8}$$

$$\begin{aligned} V^2 = \{ v^2 = (v^c, v^d) \in H^1(\Omega^c) \times H^1(\Omega^d) \mid \\ \gamma_D^c(v^c) = 0 \text{ in } H^{1/2}(\Gamma_D^c) \\ \gamma_D^d(v^d) = 0 \text{ in } H^{1/2}(\Gamma_D^d) \\ \gamma_I^c(v^c) = \gamma_I^d(v^d) \text{ in } H^{1/2}(\Gamma_I^{cd}) \} \end{aligned} \tag{9}$$

where $\gamma_D^x : H^1(\Omega^x) \to H^{1/2}(\Gamma_D^x)$ and $\gamma_I^x : H^1(\Omega^x) \to H^{1/2}(\Gamma_I^x)$. Next we introduce the constraint set consisting of those displacement fields v^α which satisfy the contact condition (6),

$$K = \{ w = (v^1, v^2) \in V^1 \times V^2 \mid \gamma_{\Sigma_n}^0(w) - g \le 0 \text{ on } \Gamma_C \} \tag{10}$$

where $\gamma_{\Sigma_n}^0 = \gamma_{\Sigma_n^1}^0 + \gamma_{\Sigma_n^2}^0$ with $\gamma_{\Sigma_n^\alpha}^0$ being the normal trace from V^α, and $\Sigma^\alpha = \text{int}\,(\Gamma^\alpha - \Gamma_D^\alpha)$, with $\bar{\Gamma}_C^\alpha \subset \Sigma^\alpha$. Let the equilibrium of each body $\bar{\Omega}^\alpha$ be characterized by the principle of virtual work; e.g.,

$$u \in V^\alpha : a^\alpha(u, v) = f^\alpha(v) + \langle p, \gamma_{\Sigma_n^\alpha}^0(v) \rangle_{\Gamma_C^\alpha} \qquad \forall\, v \in V^\alpha \tag{11}$$

where, $\langle \cdot\,, \cdot \rangle_{\Gamma_C^\alpha}$ is the duality pairing on $[H^{1/2}(\Gamma_C^\alpha)]' \times H^{1/2}(\Gamma_C^\alpha)$, and

$$a^\alpha(u, v) = \int_{\Omega^\alpha} E_{ijkm} u_{k,m} v_{i,j} \,\mathrm{d}x \tag{12}$$

$$f^\alpha(v) = \int_{\Gamma_F^\alpha} t^\alpha \cdot \gamma_{\Sigma^\alpha}^0(v) \,\mathrm{d}s \tag{13}$$

with $t^\alpha \in [H^{1/2}(\Gamma_F^\alpha)]'$. Using the bilinear form (12), a linear operator $A^\alpha : V^\alpha \to (V^\alpha)'$ can be defined such that

$$\langle A^\alpha(u), v \rangle = a^\alpha(u, v) \qquad u, v \in V^\alpha \tag{14}$$

where $\langle \cdot\,, \cdot \rangle$ denotes duality pairing on $(V^\alpha)' \times V^\alpha$. Formally, A^α is the elasticity operator whose meaning is the gradient of the strain energy functional $\tfrac{1}{2}a^\alpha(u, u)$. The operator A^α is linear, symmetric, continuous, and V^α-elliptic as shown by Kikuchi and Oden[2]. As such, it has an inverse $G^\alpha = (A^\alpha)^{-1} : (V^\alpha)' \to V^\alpha$. Furthermore, the inverse operator G^α is linear, symmetric, continuous, and positive-definite on $(V^\alpha)'$. Formally, G^α is the Green's matrix $G^\alpha(x, \eta)$ such that for any generalized force $f^\alpha \in (V^\alpha)'$ the corresponding displacement $u^\alpha \in V^\alpha$ can be written as

$$u^\alpha(x) = G^\alpha(f^\alpha) = \langle G^\alpha(x, \eta), f^\alpha(\eta)\rangle \qquad x, \eta \in \Omega^\alpha \tag{15}$$

and whose components $G^\alpha_{ij}(x, \eta)$ represent the displacements at x in the i direction due to the unit generalized force applied at η in the j direction. Here $\langle \cdot, \cdot \rangle$ denotes duality pairing on $V^\alpha \times (V^\alpha)'$.

Because the generalized force $f^\alpha \in (V^\alpha)'$ consists of two components: the boundary traction $t^\alpha \in [H^{1/2}(\Gamma^\alpha_F)]'$ and the contact pressure $p \in [H^{1/2}(\Gamma^\alpha_C)]'$, which is unknown a priori, and since the inverse operator G^α is linear on $(V^\alpha)'$, the resultant displacement u^α may be decomposed as

$$u^\alpha = \breve{u}^\alpha + \tilde{u}^\alpha \tag{16}$$

with $\breve{u}^\alpha = G^\alpha(t^\alpha)$ and $\tilde{u}^\alpha = G^\alpha(pn^\alpha)$. Similarly the vector w from $V^1 \times V^2$ may be decomposed as

$$w = \breve{w} + \tilde{w} \tag{17}$$

with $\breve{w} = (\breve{u}^1, \breve{u}^2)$ and $\tilde{w} = (\tilde{u}^1, \tilde{u}^2)$.

Using Eqs. (16) and (17), the pointwise reciprocal relation (7) leads to the variational inequality

$$p \in K^R : \langle q - p, \gamma^0_{\Sigma_n}(\tilde{w}) - \breve{g}\rangle_{\Gamma_C} \geq 0 \qquad \forall\, q \in K^R \tag{18}$$

where $K^R = \{q \in [H^{1/2}(\Gamma_C)]' \mid q \leq 0\}$, and

$$\breve{g} = g - \gamma^0_{\Sigma_n}(\breve{w}) \tag{19}$$

We refer to Eq. (18) as a reciprocal variational inequality for two-body, frictionless contact problems. Since the Green's operator G^α is positive-definite on $(V^\alpha)'$ the reciprocal variational formulation is equivalent to the following constrained minimization problem on K^R:

$$\min_{p \leq 0} \Phi(p) = \tfrac{1}{2}\langle p, \gamma^0_{\Sigma_n}(\tilde{w})\rangle_{\Gamma_C} - \langle p, \breve{g}\rangle_{\Gamma_C} \tag{20}$$

Positive-definiteness of the operator G^α guarantees a unique solution $p \in K^R$ whenever $\mathrm{mes}(\Gamma^\alpha_D) > 0$ as shown by Kikuchi and Oden.[2]

A. Analysis

Suppose that the Green's matrix (15), related to $p \in [H^{1/2}(\Gamma_C)]'$ is known, such that the displacement at $x \in \bar{\Omega}^\alpha$ due to the unit pressure applied at $\eta \in \Gamma_C$ is given by

$$\hat{u}^\alpha(x, \eta) = G^\alpha(x, \eta) \cdot n^\alpha(\eta) \tag{21}$$

so that the displacement field introduced by p may be expressed as

$$\tilde{u}^\alpha(x) = \int_{\Gamma_C} \hat{u}^\alpha(x, \eta)\, p(\eta)\, \mathrm{d}S_\eta \qquad \forall\, x \in \Omega^\alpha \tag{22}$$

Furthermore, if $\Omega^\alpha \in \mathcal{C}^{1,1}$, the normal trace of $\hat{u}^\alpha$ belongs to $H^{1/2}(\Gamma_C)$ so that the relative normal trace may be written as

$$\gamma^0_{\Sigma_n}[\tilde{w}(x)] = \int_{\Gamma_C} \gamma^0_{\Sigma_n}[\hat{w}(x, \eta)]\, p(\eta)\, \mathrm{d}S_\eta \qquad \forall\, x \in \Gamma_C \tag{23}$$

where $\hat{w} = (\hat{u}^1, \hat{u}^2) \in V^1 \times V^2$. Since the relative normal trace (23) is determined by the contact pressure, then, for a given modified gap function $\breve{g}$, the constrained minimization problem (20) is completely defined in terms of the contact pressure exerted at the surface Γ_C.

The key point of this formulation is that the minimization problem (20) has been obtained by using Green's matrix for the elastic problem without the contact condition. In mechanical terms this amounts to considering the solution of the complete problem as the superposition of two partial solutions. First is a solution for $\breve{g}$, or according to Eq. (19), a solution for the displacement fields $\breve{u}^\alpha$ in the bodies Ω^α constrained at Γ^α_D and loaded with the boundary tractions t^α along Γ^α_F. Second is a solution for Green's matrix $G^\alpha(x, \eta)$ which, according to Eq. (21), requires solving for the displacement fields $\hat{u}^\alpha$ in the bodies Ω^α constrained at Γ^α_D and loaded with unit pressure at $\eta \in \Gamma^\alpha_C$.

We define the boundary integral formulation for both partial problems and use the multiregional direct boundary element procedure for their numerical approximation.

B. Boundary Integral Equation

It is well-known that the mixed boundary value problem of finding a displacement field u in the linear elastic body $\bar{\Omega}$, with the boundary Γ, constrained at Γ_D and loaded with the boundary traction t along Γ_F, has a unique solution that admits the representation

$$\tfrac{1}{2}u(\eta) + \int_\Gamma T(x, \eta) \cdot u(x)\, \mathrm{d}\Gamma_x = \int_\Gamma U(x, \eta) \cdot t(x)\, \mathrm{d}\Gamma_x \tag{24}$$

The matrix components of U and T in Eq. (24) are given by

$$U_{mn} = \frac{1}{16\pi\mu(1-\nu)} \frac{1}{r} [(3 - 4\nu)\delta_{mn} + r_{,m}r_{,n}] \tag{25}$$

and

$$T_{mn} = -\frac{1-2\nu}{8\pi(1-\nu)} \frac{1}{r^2} \left[\frac{\partial r}{\partial n} \left(\delta_{mn} + \frac{3}{1-2\nu} r_{,m}r_{,n} \right) - n_n r_{,m} + n_m r_{,n} \right] \tag{26}$$

with $1 \leq m, n \leq 3$. Here λ and μ are Lame coefficients, and $\nu = \lambda/2(\lambda - \mu)$ is the Poisson ratio of a body Ω. In addition, r is the distance between the field point x and the load point η; note that all differentiations are with respect to the field point.

Equation (24) can be viewed as a constrained equation relating surface tractions to surface displacements. We establish an integral equation (24) for each of the displacements $\breve{u}^\alpha(x)$ and $\hat{u}^\alpha(x, \eta)$ and use the direct boundary element procedure for their numerical approximation.

IV. Numerical Solution of the Problem

A boundary element technique was adopted as a method for discretization of the continuous variational problem (20), leading to the finite-dimensional quadratic programming problem. Boundary integral equations for the displacements $\breve{u}^\alpha$ and $\hat{u}^\alpha$ are approximated using a standard direct multiregional boundary element procedure.

A. Approximation of the Variational Inequality

We partition the candidate contact surface Γ_C into M_C flat polygonal regions $\{\Gamma^i_C\}^{M_C}_{i=1}$. We approximate the smooth function $p \in H^{-1/2}(\Gamma_C)$ by its finite-dimensional subspace of all linear combinations of the zero-order shape functions which are constant over any boundary element Γ^i_C in Γ_C, namely,

$$p(x) = \sum_{i=1}^{M_C} \phi_i(x) p_i \tag{27}$$

where $p_i \leq 0 \in \mathbb{R}$, and $\phi_i(x)$ is an indicator function for the boundary element Γ^i_C;

$$\phi_i(x) = \begin{cases} 1 & \text{if } x \in \Gamma^i_C \\ 0 & \text{if } x \notin \Gamma^i_C \end{cases} \tag{28}$$

For a chosen approximation of p, the relative normal trace, Eq. (23), can be computed as

$$\gamma^0_{\Sigma_n}[\tilde{w}(x)] = \sum_{j=1}^{M_C} \gamma_j(x) p_j \tag{29}$$

with

$$\gamma_j(x) = \gamma^0_{\Sigma_n}[\hat{w}_j(x)] \tag{30}$$

where $\hat{w}_j(x) = [\hat{u}^1_j(x), \hat{u}^2_j(x)]$ and

$$\hat{u}^\alpha_j(x) = \int_{\Gamma_C} \hat{u}^\alpha(x, \eta)\phi_j(\eta)\, \mathrm{d}S_\eta \tag{31}$$

Comparison of Eqs. (22) and (31) show that $\hat{u}_j^\alpha(x)$ is a displacement at x in the body Ω^α constrained at Γ_D^α and loaded with a unit pressure along Γ_C^j.

Using the approximations for p and $\gamma_{\Sigma_n}^0$, the minimization problem (20) becomes the following finite-dimensional quadratic programming problem,

$$\min_{p \in K^{M_C}} \Phi_{M_C}(p) = \frac{1}{2} \sum_{i=1}^{M_C} \sum_{j=1}^{M_C} A_{ij} p_i p_j - \sum_{i=1}^{M_C} B_i p_i \tag{32}$$

where $K^{M_C} = \{ p \in \Re^{M_C} | \forall\ i\ p_i \leq 0 \}$ and

$$A_{ij} = \int_{\Gamma_C} \phi_i(x) \gamma_j(x)\, dS_x \simeq \text{mes}(\Gamma_C^i)\gamma_j(x_i) \tag{33}$$

$$B_i = \int_{\Gamma_C} \phi_i(x) \breve{g}(x)\, dS_x \simeq \text{mes}(\Gamma_C^i)\breve{g}(x_i) \tag{34}$$

where, the functions $\gamma_j(x)$ and $\breve{g}(x)$ have been approximated by the piecewise step functions, i.e.,

$$\gamma_j(x_i) = n(x_i) \cdot [\hat{u}_j^1(x_i) - \hat{u}_j^2(x_i)] \tag{35}$$

$$\breve{g}(x_i) = g(x_i) - n(x_i) \cdot [\breve{u}^1(x_i) - \breve{u}^2(x_i)] \tag{36}$$

Thus the problem becomes one of finding the displacements $\breve{u}^\alpha(x_i)$ and $\hat{u}_j^\alpha(x_i)$ for $\alpha = 1$, 2 and $i, j = 1, \ldots, M_C$. As mentioned in Sec. III.A. we utilize a direct multiregional boundary element method for the numerical evaluation of these displacements.

B. Boundary Element Formulation

Consider the numerical approximation for the displacement fields $\breve{u}^\alpha$. We discretize the boundary Γ^x of each constituent body Ω^x into M^x plane triangular elements, $\{\Gamma^l\}_{l=1}^{M^x}$, such that $\{\Gamma_C^i\} \subset \{\Gamma^l\}$. Let the boundary data be approximated by

$$\breve{u}^x(x) = \sum_{l=1}^{M^x} \phi_l(x)\, \breve{u}^x(x_l) \tag{37}$$

$$t^x(x) = \sum_{l=1}^{M^x} \phi_l(x)\, t^x(x_l) \tag{38}$$

where $\breve{u}^x(x_l)$, $t^x(x_l) \in \Re^3$ are displacement and traction at the centroid x_l of the boundary element Γ^l and the indicator function for the boundary element Γ^l is defined as in Eq. (28). Substituting these functions into Eq. (24) written for the centroid η_k of the boundary element Γ^k, we arrive at a linear algebraic system of the form

$$\frac{1}{2}\breve{u}^x(\eta_k) + \sum_{l=1}^{M^x} T_{lk}^x \cdot \breve{u}^x(x_l) = \sum_{l=1}^{M^x} U_{lk}^x \cdot t^x(x_l) \tag{39}$$

The integrals

$$T_{lk}^x = \int_{\Gamma^x} \phi_l(x) T^x(x, \eta_k)\, d\Gamma_x \tag{40}$$

$$U_{lk}^x = \int_{\Gamma^x} \phi_l(x) U^x(x, \eta_k)\, d\Gamma_x \tag{41}$$

are calculated exactly from the size, orientation, and location of the element Γ^l and the point η_k, according to Cruse.[13] Combining the Eqs. (39), written for the bodies Ω^a and Ω^b, with the displacement and traction continuity boundary conditions (4) over the internal boundary Γ_I^{ab}, one obtains the set of algebraic equations

$$A^\alpha \cdot X^\alpha = B^\alpha \tag{42}$$

related to the body $\alpha = 1$. Unknown data in Eq. (42) are the displacements $\breve{u}^1$ along int$(\Gamma^1 - \Gamma_D^1)$, traction along Γ_D^1, and the tractions and displacements along the interface Γ_I^{ab}. Solving Eq. (42) one can obtain the boundary displacements $\breve{u}^1(x_i)$ in the candidate contact surface $\{\Gamma_C^i\}_{i=1}^{M_C}$. Following the same procedure, but now for the constituent bodies Ω^c and Ω^d, an equation similar to Eq. (42) for the body $\alpha = 2$ can be obtained. Its solution provides the displacements $\breve{u}^2(x_i)$ in the candidate contact surface. Knowing the displacements $\breve{u}^\alpha(x_i)$ the approximation for the modified gap function $\breve{g}(x_i)$ can be obtained from Eq. (36).

Consider now the numerical approximation for the displacement fields $\hat{u}_j^\alpha (j = 1, \ldots, M_C)$, as defined by Eq. (31). To apply integral equation (24) to this case, the boundary traction is set as

$$t_j^\alpha(x) = \phi_j(x) n^\alpha(x) \tag{43}$$

Now the problem becomes one of finding displacements in the body Ω^α constrained at Γ_D^α and loaded with the unit pressure along $\Gamma^j \in \Gamma_C$. Applying the procedure used in establishing Eq. (42), we obtain

$$A^\alpha \cdot X_j^\alpha = B_j^\alpha \tag{44}$$

where the matrices A^α are precisely those defined by Eq. (42). Solving Eq. (44), one obtains the boundary displacements $\hat{u}_j^\alpha(x_i)$ in the candidate contact surface $\{\Gamma_C^i\}_{i=1}^{M_C}$. To establish Eq. (35) for all j elements in Γ_C Eqs. (44) must be solved $2 \times M_C$ times, with prescribed boundary conditions on Γ_D^α and Γ^j. The fact that A^α matrices in Eqs. (42) and (44) are equal allows for an efficient numerical solution of the problem involving only one matrix inversion per body.

C. Algorithm

We now define an algorithm for the solution of the quadratic programming problem (32):

1) Discretize the boundaries Γ^x of each constituent body Ω^x into M^x plane triangular elements and calculate the integrals T_{lk}^x and U_{lk}^x using Eqs. (40) and (41), respectively.

2) Using the matrices T_{lk}^x, U_{lk}^x, with the given essential and interfacial boundary conditions, assemble and invert the matrices $A^\alpha (\alpha = 1, 2)$.

3) Using the matrices T_{lk}^x, U_{lk}^x, with the given natural boundary conditions assemble the B^α vector and obtain the solution vector by multiplying $(A^\alpha)^{-1} \cdot B^\alpha$ $(\alpha = 1, 2)$. Extract the displacements $\breve{u}^\alpha(x_i)$ in the candidate contact surface $\{\Gamma_C^i\}_{i=1}^{M_C}$ from the solution vector X^α.

4) For a given gap function $g(x_i)$ and displacements $\breve{u}^\alpha(x_i)$, obtain the modified gap function $\breve{g}(x_i)$, using Eq. (36), and the linear part of the functional $\Phi_{M_C}(p)$, B_i, using Eq. (34).

5) Using the matrices T_{lk}^x, U_{lk}^x, assemble the vectors B_j^α from Eq. (44) and obtain the solution $(A^\alpha)^{-1} \cdot B_j^\alpha$ for all j elements in Γ_C. Extract the displacements $\hat{u}_j^\alpha(x_i)$ in the candidate contact surface $\{\Gamma_C^i\}_{i=1}^{M_C}$ from the solution vector X_j^α.

6) Knowing the displacements $\hat{u}_j^\alpha(x_i)$, calculate the normal trace $\gamma_j(x_i)$ using Eq. (35), and then the quadratic part of the functional $\Phi_{M_C}(p)$, A_{ij}, from Eq. (33).

7) Solve the quadratic programming problem (32).

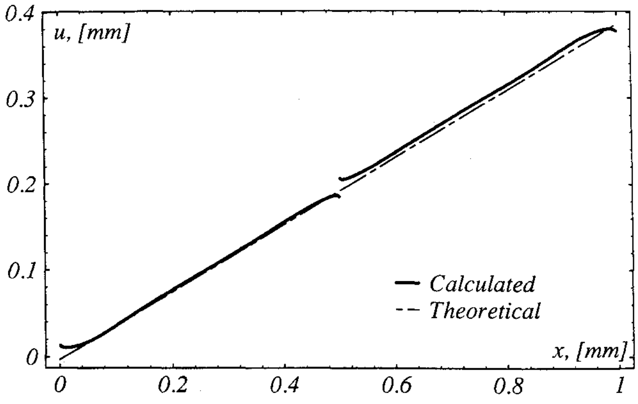

Fig. 2 Axial displacements vs axial distance for the homogeneous unit cube.

V. Applications

Several boundary value problems were solved to verify the accuracy of the approximations discussed in the previous sections. The computed results are given for a unit cube. In all cases sufficient displacements were set to zero, i.e., mes$(\Gamma_D^\alpha) > 0$, $(\alpha = 1, 2)$, to eliminate rigid-body motion. Required data includes the material properties, the surface element arrangement, the known surface traction and displacements (which are assumed constant over each surface element), and the locations of internal points where the displacements and stresses are required. In the case of a rigid-punch problem the total compressive force, or depth of indentation, is required. The results consist of the contact area, the pressure acting in it, as well as stresses and displacements where required.

A. Boundary Element Applications

First, the basic assumptions related to Green's function estimation have been checked. To that purpose a classical multiregion boundary-element-method-type of calculation has been performed without the complications of contact conditions. In the first experiment the elastic homogeneous unit cube was subdivided into two equal zones and loaded in a state of uniaxial tension by the application of the unit normal traction to one face which is parallel to the interface. Furthermore, on the opposite face and on two other faces, the normal displacements were set equal to zero. The displacements u along the loading axis were calculated and compared against an analytical solution. The results for the case of 192 boundary elements with Lame coefficients $\lambda = 1.5$ and $\mu = 1$ are given in Fig. 2. Next, material properties of the second subregion were changed; i.e., μ was changed to 0.6. The comparison of the numerical calculation with the analytical results is given in Fig. 3. In both instances, the numerical calculations are in good agreement with the analytical results.

Similar calculations were performed for the unit cube subjected to a uniform shear stress. The results obtained are in agreement with the analytical results and they are omitted here for brevity.

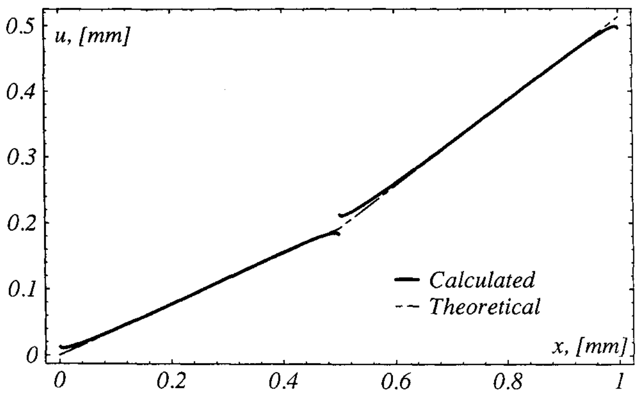

Fig. 3 Axial displacements vs axial distance for the inhomogeneous unit cube.

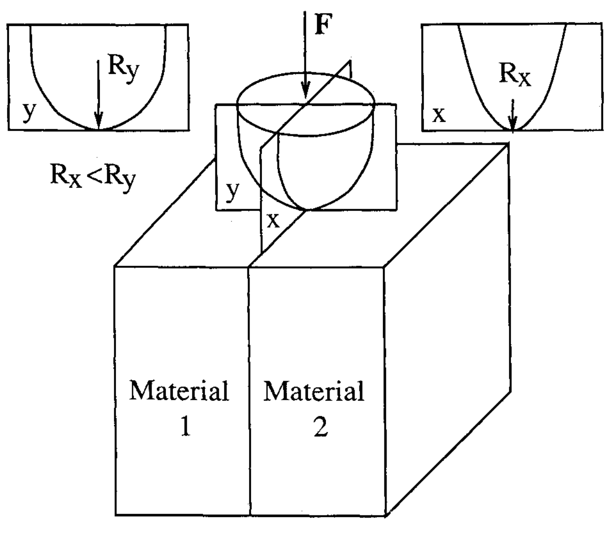

Fig. 4 Indentation of an elastic unit cube by a rigid punch with the shape of second-degree paraboloid.

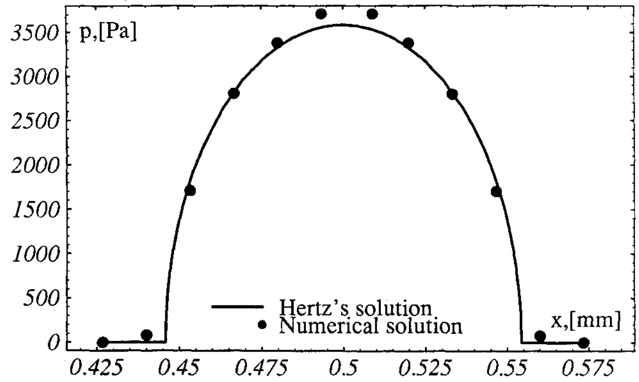

Fig. 5 Contact pressure in x plane of Fig. 4.

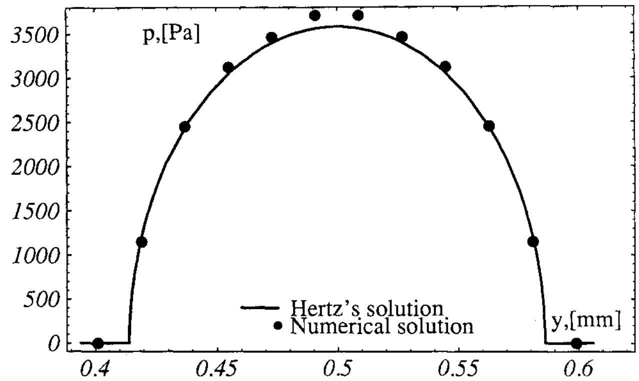

Fig. 6 Contact pressure in y plane of Fig. 4.

B. Half-Space Contact Problems

To estimate the accuracy of the approximations related to the contact problem, we consider the indentation of a rigid, frictionless punch into an elastic, homogeneous unit cube, as shown in Fig. 4. The cube, although homogeneous, is subdivided into two regions with the same material properties. The shape of the punch is a second-degree paraboloid. This is a well-known three-dimensional contact problem which in the case of a semi-infinite foundation has analytical solutions; namely, the Hertz problem (see Johnson[14]). It may be expected that the influence of the free sides of the cube will be negligible in this case, yielding a solution that will be close to the analytical one.

The results presented here are for an indentation depth $d = 1.66 \times 10^{-3}$, and for the case of $12 \times 12 \times 2$ triangles in the candidate contact zone. The total number of elements was 712 with 388 interfacial elements. The contact pressure in the planes x and y of Fig. 4 are presented in Figs. 5 and 6, respectively.

The numerical results are plotted along with the analytical solutions of Hertz for the half-space. The fact that the results of the numerical calculations are in good agreement with the analytical solution prove the ability of our method to deal with this type of contact problem and indicates that the influence of the free boundary is negligible.

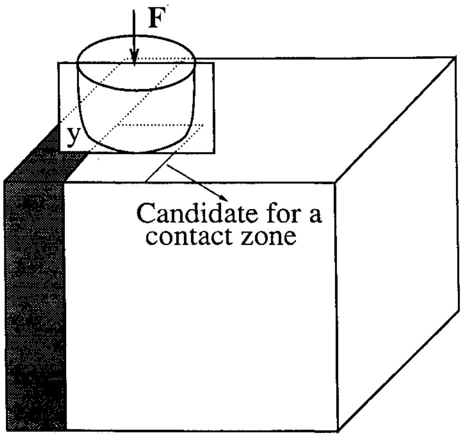

Fig. 7 Indentation of a corner of a coated elastic unit cube by a rigid punch.

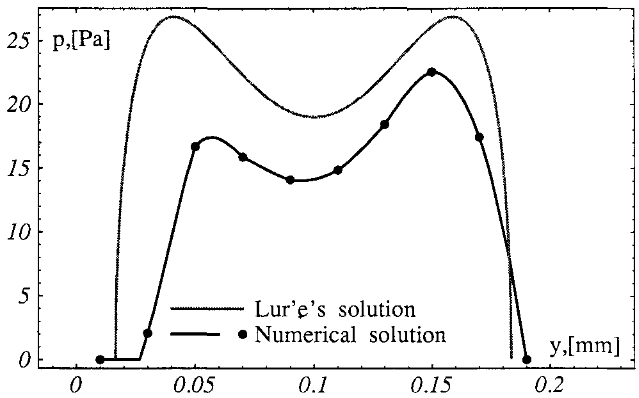

Fig. 8 Contact pressure in *y* plane of Fig. 8 for a homogeneous uncoated cube.

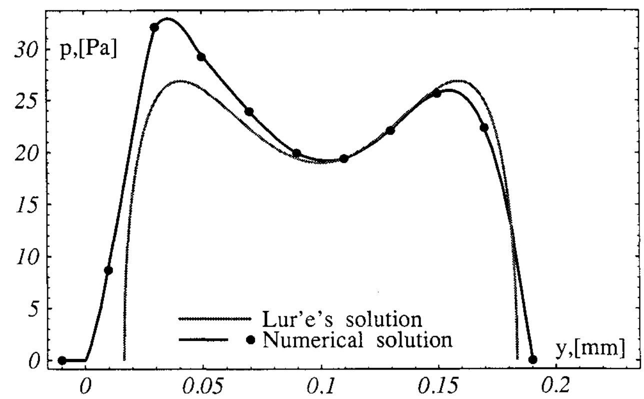

Fig. 9 Contact pressure in *y* plane of Fig. 8. Illustration of the influence of the inhomogenity.

C. Finite Bodies Contact Problems

To examine the free boundary effect, we have performed a series of calculations for the axisymmetrical rigid punch with the shape of a fourth-order paraboloid. The punch is indenting at the corner of the cube as shown in Fig. 7.

In the first experiment the unit cube was not coated. The contact pressure in the y plane of Fig. 7 is presented in Fig. 8. The results presented are for the depth of indentation $d = 1.64 \times 10^{-5}$, and for $10 \times 10 \times 2$ triangles in the candidate contact zone. The total number of elements was 798 with 384 interfacial elements. For comparison, numerical results are plotted along with the analytical solution for the half-space problem (see Lur'e[15]). One can see that the pressure is lower at the points close to the free surface. This might be anticipated as the material is "softer" close to the free boundary.

If we repeat the experiment with a hard-coated cube, the contact pressure increases at the points close to the coating as shown in Fig. 9.

VI. Conclusion

In this paper a method for the numerical solution of the frictionless contact between two elastic composite bodies having arbitrary shape and loading is presented. Starting from a rigorous variational formulation for the contact problem and its related extremum principle, a numerical method is defined by means of a multiregional boundary element discretization. More specifically, the Green's function associated with the formulation is approximated by means of a standard, multiregional, direct boundary element procedure. The discretization leads to the finite-dimensional quadratic programming problem which was solved by a modification of the gradient-projection method.

A variety of problems including Hertzian and non-Hertzian contact problems were investigated numerically. The results of our calculations are in good agreement with existing analytical results. Presented here are important new results which illustrate the distinct ability of the method to capture the influence of the composite structure of a body on the contact area and the pressure acting in it.

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